Critical behavior of random fibers with mixed Weibull distribution

Uma Divakaran* and Amit Dutta†
Department of Physics, Indian Institute of Technology Kanpur 208016, India
(Received 2 August 2006; published 11 January 2007)

A random fiber bundle model with a mixed Weibull distribution is studied under the global load sharing scheme. The mixed model consists of two sets of fibers. The threshold strength of one set of fibers is randomly chosen from a Weibull distribution with a particular Weibull index, and another set of fibers with a different index. The mixing tunes the critical stress of the bundle and the variation of critical stress with the amount of mixing is determined using a probabilistic method where the external load is increased quasistatically. In a special case which we illustrate, the critical stress is found to vary linearly with the mixing parameter. The critical exponents and power-law behavior of burst avalanche size distribution is found to remain unaltered due to mixing.

DOI: 10.1103/PhysRevE.75.011109 PACS number(s): 64.60.Ht, 46.50.+a, 62.20.Mk, 81.05.Ni

I. INTRODUCTION

Sudden catastrophic failure of structures due to unexpected fracture of component materials is a concern and a challenging problem of physics as well as engineering. The dynamics of the failure of materials show interesting properties and hence there has been an enormous amount of study on the breakdown phenomena up to now [1–3]. The complexity involved in fracture processes can be suitably modeled by grossly simplified models. The simplest available model is fiber bundle model (FBM) [4–7].

A FBM consists of N parallel fibers. The disorder of a real system is introduced in the fiber bundle in the form of random distribution of strength of each fiber taken from a probability density $p(\sigma)$ and hence called random fiber bundle model (RFBM). The strength of each fiber is called its threshold strength. As a force F is applied externally on a bundle of N fibers, a stress $\sigma = F/N$ develops on each of them. The fibers which have their threshold strength smaller than the stress generated will break immediately. The next question that arises is the affect of breaking of these fibers on the remaining intact fibers, i.e., one has to decide a load sharing rule. The two extreme cases of load sharing mechanisms are global load sharing (GLS) [5,8,9] and local load sharing (LLS) [10–12]. In GLS, the stress of the broken fiber is equally distributed to the remaining intact fibers. This rule neglects local fluctuations in stress and therefore is effectively a mean field model with long range interactions among the elements of the system [13]. On the other hand, in LLS the stress of the broken fiber is given only to its nearest surviving neighbors. It is obvious that the actual breaking process involves a sharing rule which is in between GLS and LLS. Several studies have been made which consider a rule interpolating between GLS and LLS [14,15].

For a given force F, some fibers break and they distribute their load to the surviving fibers following a load sharing rule causing further failures and redistribution of stress. This process continues until all the remaining fibers have their threshold strength greater than the redistributed stress acting on them. This corresponds to the fixed point of the dynamics of the system. As the applied force is increased on the system, more and more fibers break. An avalanche of size Δ is defined as the number of failed fibers between two successive external loadings. There exists a critical load (or stress σ_c) beyond which if the load is applied, complete failure of the system takes place. Most of the studies on FBM involve the determination of the critical stress σ_c and the investigation of the type of phase transition from a state of partial failure to a state of complete failure. It has been shown that a bundle following GLS has a finite value of critical stress and belongs to a universality class with a specific set of critical exponents [8,16] whereas there is no finite critical stress σ_c at thermodynamic limit in the case of LLS in one dimension [10,18]. On the other hand, LLS on complex network has been shown to belong to the same universality class as that of GLS with the same critical exponents [19].

In this paper, we study a FBM with mixed fibers. Fibers have their threshold strength randomly chosen from Weibull distributions with two different index parameters. The motivation here is to study the dynamics of random fibers in the presence of disorder caused due to mixing of two types of fibers with overlapping distribution of threshold strengths. Moreover, the probabilistic method implemented to estimate the critical stress can be very easily used to study any type of mixed fiber bundles thus enabling one to put maximum disorder in the system being studied.

Section II consists of description of the model highlighting the method used. In Sec. III we present the results. Section IV includes discussions and conclusions.

II. THE MODEL

In the Weibull distribution (WD) of threshold strength of fibers, the probability of failure of each element when a stress σ is generated has a form

$$P(\sigma) = 1 - e^{-(\sigma/\sigma_0)^\rho}, \tag{1}$$

where σ_0 is a reference strength and ρ is called the Weibull index. We consider a mixed RFBM with WD of threshold

*Electronic address: udiva@iitk.ac.in
†Electronic address: dutta@iitk.ac.in

strength of fibers where strengths of fibers are randomly chosen from two different distributions characterized by different Weibull indices and study the critical behavior of the model. A fraction x (henceforth called the mixing parameter) of fibers belong to the class A (WD with $\rho=\rho_1$) and the remaining $(1-x)$ fraction belong to the class B (WD with $\rho=\rho_2$) with the reference strength σ_0 set equal to 1 in both the distributions.

The probabilistic method introduced by Moreno, Gomez, and Pacheco [9] is extended to explore the critical behavior of the above model. For the conventional WD ($x=1$ or 0), let us consider a situation where the stress on each fiber increases from σ_1 to σ_2. The probability that a fiber randomly chosen from the WD survives from the load σ_1 but fails when the load is σ_2, is given by

$$p(\sigma_1,\sigma_2) = \frac{P(\sigma_2) - P(\sigma_1)}{1 - P(\sigma_1)} = 1 - e^{-(\sigma_2^\rho - \sigma_1^\rho)}.$$

Thus the probability that the chosen fiber that has survived the load σ_1 also survives the (higher) load σ_2 is $q(\sigma_1,\sigma_2) = e^{-(\sigma_2^\rho-\sigma_1^\rho)}$. The key point is that the force F on the bundle is increased quasistatically so that only the weakest fiber amongst the remaining intact fibers breaks. Thus, one needs to identify the weakest fiber, break that fiber, and then calculate the number of remaining unbroken fibers using $q(\sigma_1,\sigma_2)$ recursively, updating the value of σ_2 and σ_1 due to the load given away by the broken fibers, until no more failure occurs. As mentioned previously, the dynamics of breaking will continue until the system reaches a fixed point. The process of slow increase of external load is carried on up to the critical stress σ_c. The method avoids the random averaging involved in a Monte Carlo simulation and hence turns out to be very useful in dealing with fluctuations.

However, to deal with the "mixed" random fiber bundle model, the above method needs to be generalized in an appropriate manner. The essential point is the fact that we need to keep track of number of unbroken fibers in each distribution separately. Let us assume that after a loading is done and a fixed point is reached, N_{k_1} and N_{k_2} are the number of unbroken fibers corresponding to $\rho=\rho_1$ and $\rho=\rho_2$ distribution, respectively, and σ_k is the stress per fiber at that instant.

We define q_1 and q_2 as follows:

$$q_1(\sigma_1,\sigma_2) = e^{-(\sigma_2^{\rho_1}-\sigma_1^{\rho_1})},$$

$$q_2(\sigma_1,\sigma_2) = e^{-(\sigma_2^{\rho_2}-\sigma_1^{\rho_2})}.$$

One needs to calculate the load $(N_{k_1}+N_{k_2})\sigma_l$ that has to be applied to break one fiber. Here, $\sigma_l=\min(\sigma_{l_1},\sigma_{l_2})$ where σ_{l_1} (or σ_{l_2}) is the next weakest fiber in the $\rho=\rho_1$ (or $\rho=\rho_2$) distribution of strength of fibers and is obtained by the solution of the following equations:

$$N_{k_1} - 1 = N_{k_1} q_1(\sigma_k,\sigma_{l_1}),$$

and

$$N_{k_2} - 1 = N_{k_2} q_2(\sigma_k,\sigma_{l_2})$$

which gives

$$\sigma_{l_1} = \left[\sigma_k^{\rho_1} - \ln\left(1 - \frac{1}{N_{k_1}}\right)\right]^{1/\rho_1}, \tag{2}$$

$$\sigma_{l_2} = \left[\sigma_k^{\rho_2} - \ln\left(1 - \frac{1}{N_{k_2}}\right)\right]^{1/\rho_2}. \tag{3}$$

The breaking of one fiber and the redistribution of its stress to all the remaining intact fibers causes some more failures. Let us assume that during this avalanche, at some point before the fixed point is reached, there are $\tilde{N}_{k_1}$ and $\tilde{N}_{k_2}$ number of unbroken fibers belonging to the two distributions where each fiber is under a stress $\tilde{\sigma}_k$. This stress causes some more failures and as a result N'_{k_1} and N'_{k_2} fibers are unbroken. Let the new stress developed be σ'_k. The number of fibers which survive $\tilde{\sigma}_k$ and σ'_k are obtained using the relation

$$N''_{k_1} = N'_{k_1} q_1(\tilde{\sigma}_k,\sigma'_k), \tag{4}$$

$$N''_{k_2} = N'_{k_2} q_2(\tilde{\sigma}_k,\sigma'_k). \tag{5}$$

The stress on each fiber is now equal to $(N'_{k_2}+N'_{k_1})\sigma'_k/(N''_{k_1}+N''_{k_2})$. Equations (4) and (5) are used again and again until a fixed point is reached. The fixed point condition is given by $(N'_{k_1}+N'_{k_2})-(N''_{k_1}+N''_{k_2})<\epsilon$ where ϵ is a small number (0.001). It should be noted that the critical behavior does not depend on the choice of ϵ. After the fixed point is reached, stress σ_l is calculated once again as mentioned before and the whole process is repeated until complete failure occurs at σ_c.

It should also be mentioned here that the critical stress σ_c can also be derived using directly the cumulative distribution function for the mixed model as given in Eq. (6) in $p(\sigma_1,\sigma_2)$. However, in this case, the expression of σ_l, as defined above, turns out to be very complicated and it is difficult to arrive at a simple closed form as shown in Eq. (2) or (3).

III. RESULTS

We present here the main results of a particular case, $x=0.5$, $\rho_1=2$, and $\rho_2=3$. Figure 1 shows the fraction of total number of broken fibers as a function of applied stress σ. The graph clearly shows the existence of a critical stress $\sigma_c=0.46$ at which fraction of failed fibers increases rapidly and the bundle breaks down completely. The critical stress of the mixed bundle lies between that of the two pure bundles [for $\rho=2$, $\sigma_c=0.42$ and for $\rho=3$, $\sigma_c=0.49$ obtained using $\sigma_c=(\rho e)^{-1/\rho}$] [5]. Thus, the resulting critical stress of the mixed fiber bundle model can be tuned by varying the mixing parameter x.

The mean avalanche size S of failure is defined as the total number of broken fibers between two successive loadings. It diverges near the critical point as $(\sigma_c-\sigma)^{-\gamma}$ with an exponent $\gamma=1/2$. Scaling behavior of S is shown in Fig. 2.

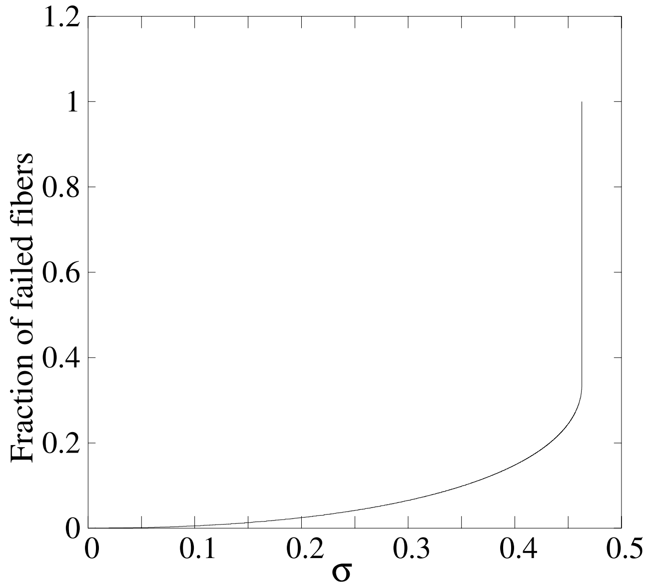

FIG. 1. Variation of fraction of broken fibers with external load per fiber σ for a RFBM with two distributions corresponding to $\rho_1=2$, $\rho_2=3$, $x=0.5$, and $N=50\,000$.

The important feature associated with the failure process in RFBM is the power-law behavior of burst avalanche distribution of fibers. The probabilistic approach [9] implemented to determine σ_c turns out to be inappropriate in exploring the behavior of avalanche size distribution. This is due to the fact that the mean avalanche sizes (S) obtained for different σ are of fractional values which leads to a difficulty in calculating the distribution of avalanche sizes. We therefore use the standard Monte Carlo method along with the weakest fiber approach [17] where the external load is increased by an amount sufficient to break the weakest intact fiber. The corresponding integer value of the size of an avalanche is denoted by Δ. For GLS, the distribution $D(\Delta)$ of an

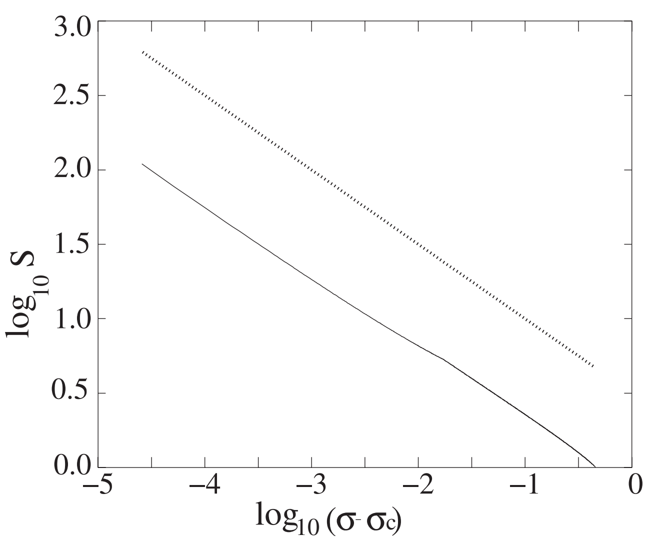

FIG. 2. Scaling behavior of mean avalanche size S as the critical point is reached for the mixed model. Also shown is a straight line (dotted) with slope $(-1/2)$. Here, $N=50\,000$ and $x=0.5$.

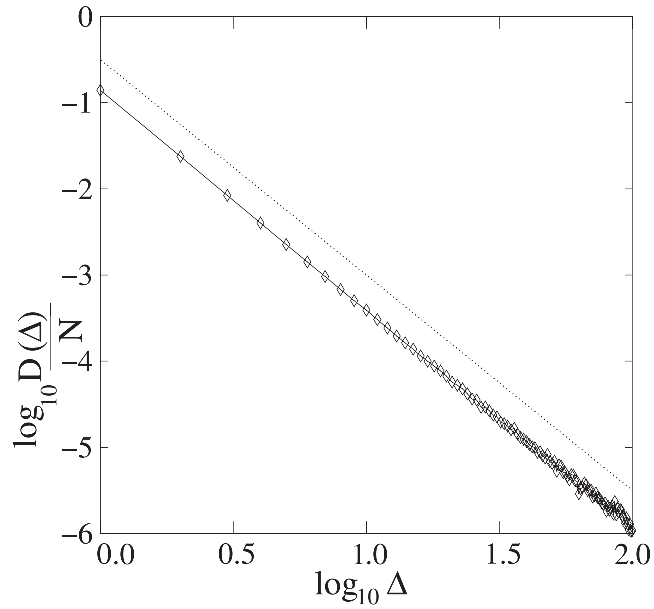

FIG. 3. Avalanche size distribution for $x=0.5$, $\rho_1=2$, $\rho_2=3$, and $N=50\,000$. A straight line with slope $-5/2$ is also shown.

avalanche of size Δ follows a power law $D(\Delta)\propto\Delta^{-\xi}$, where $\xi=5/2$ in the asymptotic limit [17]. Figure 3 shows the avalanche size distribution for the present mixed RFBM obtained numerically with $x=0.5$; a power-law behavior with the same exponent $5/2$ is clearly observed, confirming the mean field nature of the model. That the $5/2$ behavior is expected even for a mixed RFBM for any x can be justified using the saddle point method applicable in the limit of large Δ [17]. In the present case ($\rho_1=2$ and $\rho_2=3$), the probability that a fiber will break when subjected to a stress σ is

$$P(\sigma)=x[1-\exp(-\sigma^2)]+(1-x)[1-\exp(-\sigma^3)] \quad (6)$$

so that the density distribution becomes

$$p(\sigma)=2x\sigma\exp(-\sigma^2)+(1-x)3\sigma^2\exp(-\sigma^3). \quad (7)$$

The avalanche size distribution takes the form

$$\frac{D(\Delta)}{N}=\frac{\Delta^{\Delta-1}}{\Delta!}\int_0^{\sigma^*} d\sigma\frac{1}{\sigma}[1-P(\sigma)-\sigma p(\sigma)] \times\left[\frac{\sigma p(\sigma)}{1-P(\sigma)}\exp\left(-\frac{\sigma p(\sigma)}{1-P(\sigma)}\right)\right]^{\Delta}, \quad (8)$$

where σ^* is the redistributed stress at the critical point at which the average applied force $[F=N\sigma(1-P(\sigma))]$ maximizes, and $p(\sigma)$ and $P(\sigma)$ are as defined above. The function inside the square bracket has a maximum when

$$\frac{\sigma p(\sigma)}{1-P(\sigma)}=1. \quad (9)$$

It should be noted that the above condition is satisfied when $\sigma=\sigma^*$. Since the threshold distribution of fibers [Eq. (7)] does not have any discontinuity, the saddle point integration of Eq. (8) (retaining the first-order term in the expansion of

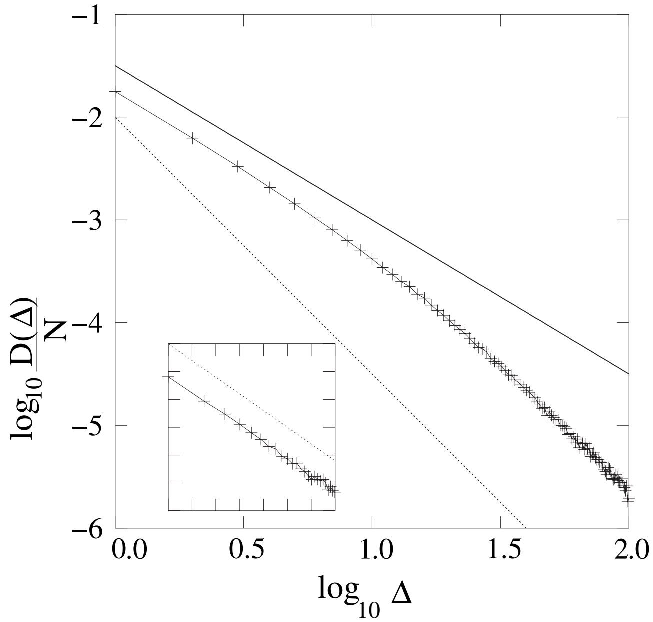

FIG. 4. The variation of $D(\Delta)$ with Δ when close to the critical distribution for a mixed Weibull distribution with $\rho_1=2$, $\rho_2=3$, and $x=0.5$. A clear crossover from $\Delta^{-3/2}$ to $\Delta^{-5/2}$ is observed with $\sigma_0=0.62$. The slope of the thick line is $-3/2$ and dotted line is $-5/2$. The inset shows the avalanche size distribution for the critical case along with a line of slope of $-3/2$.

the prefactor $[1-P(\sigma)-\sigma p(\sigma)]$ around $\sigma=\sigma^*$) yields the asymptotic behavior $D(\Delta)\propto\Delta^{-5/2}$.

Let us now comment on the imminent failure [20] behavior, when a fraction of weak fibers are already removed, and the distribution is close to the critical distribution, i.e., the strength of the weakest intact fiber (σ_0) is close to the redistributed stress σ^* at the critical point. We shall consider the case when $\rho_1=2$ and $\rho_2=3$. The variation of $D(\Delta)$ with Δ is shown in Fig. 4 and as in the pure Weibull case, we observe a crossover from $\Delta^{-3/2}$ to $\Delta^{-5/2}$ as Δ increases. For the critical distribution, however, we observe a $\Delta^{-3/2}$ behavior for the whole range of Δ (inset Fig. 4).

Although the power law behavior of the avalanche size distribution near the critical distribution remain unaltered by mixing, one may ask the question about the behavior of Δ_c with x (where Δ_c denotes the avalanche size at which crossover from 3/2 to 5/2 is observed). For the case of interest, $\rho_1=2$ and $\rho_2=3$, Δ_c does not change appreciably with x. This is due to the fact that σ^* is almost constant as x is varied (explained later). On the other hand, if we consider the case $\rho_1=1$ and $\rho_2=3$ (Fig. 6), Δ_c should decrease with x keeping σ_0 constant because as x is increased σ^* also increases taking the system away from critical distribution.

The critical exponent γ and the power-law behavior of avalanche size distribution of the mixed model remains unchanged for any value of mixing parameter x. This supports the fact that the critical behavior of a fiber bundle model is determined entirely by the load-sharing rule.

However, the variation of σ_c with the mixing parameter x shows a very interesting linear behavior (Fig. 5). To justify the linear behavior, we recall the probability distribution [Eq. (6)]. A relation between the critical stress and σ^* can be

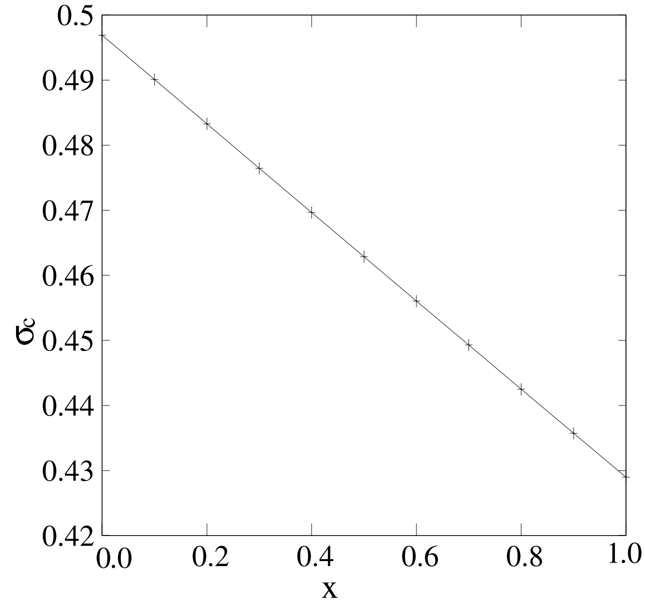

FIG. 5. Variation of critical stress σ_c with x. Slope of the straight line is 0.068. $N=50\,000$.

obtained by calculating the applied force at σ^* where σ^* is a solution of Eq. (9):

$$\sigma_c=\frac{F}{N}=\sigma^{*2}p(\sigma^*). \tag{10}$$

For the conventional WD, $\sigma^*=(\frac{1}{\rho})^{1/\rho}$, which immediately reveals the fact that the difference in σ^* for the cases $\rho=2$ and $\rho=3$ is very small as compared to the change in corresponding σ_c values. To a good approximation, one can

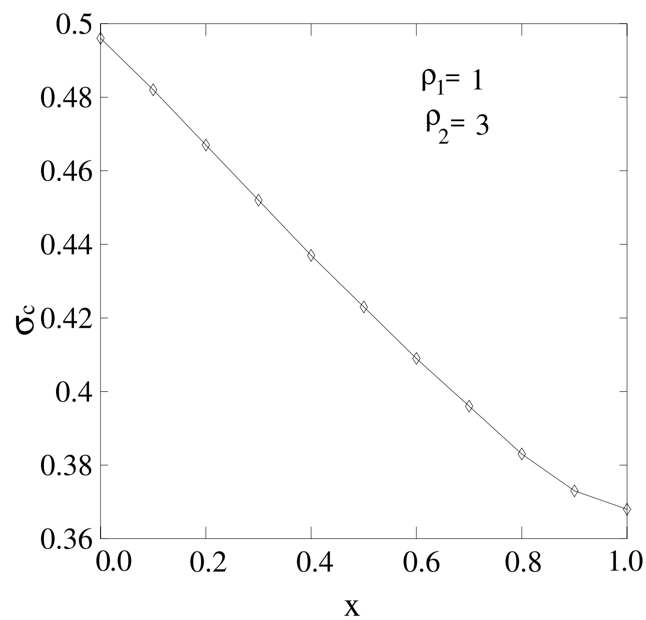

FIG. 6. Nonlinear variation of critical stress with x for $\rho_1=1$ and $\rho_2=3$.

set σ^*=constant=c, so that Eq. (10) combined with Eq. (7) yields

$$\sigma_c = c^2[2xc\exp(-c^2)] + (1-x)3c^2\exp(-c^3)$$
$$= xc^2[2c\exp(-c^2) - 3c^2\exp(-c^3)] + 3c^4\exp(-c^3),$$

which justifies the dominant linear dependence of σ_c on x for this particular case. It should also be emphasized that this linear relationship is a characteristic of situations where the variation of σ^* is negligible (to the lowest order) as x is tuned from 0 to 1 (e.g., the present case) and in general there is a nonlinear relationship as shown in Fig. 6.

In a recent work [21], the shear failure of a glued interface has been studied using a simple beam model where beams (fibers) connect the two surfaces. The stretching and bending threshold strength of the beams, denoted by ϵ_1 and ϵ_2, respectively, are randomly distributed variables satisfying a joint probability distribution function $p(\epsilon_1, \epsilon_2)$. The mean field critical exponents are obtained when the threshold distributions for bending and stretching modes are independent and chosen from two different WD. In our model, we study fibers with threshold strength chosen from two Weibull distributions with different Weibull indices and the critical exponents stick to the mean field values also in our case.

IV. CONCLUSION

The critical stress of a mixed RFBM with Weibull distribution and GLS is studied using a probabilistic approach where the external force is increased quasistatically at every step of loading [9]. The advantage of this method is that it does not require the process of random averaging which takes comparatively longer computational time. We obtain the variation of critical stress with parameter x. This functional dependence of the critical stress on the characteristic quantity "the mixing parameter" is an important objective of this work.

The critical behavior of the mixed model namely the critical exponents and the power law behavior of the burst avalanche distribution are the same as the mean field. In a mixed Weibull distribution, the threshold distributions of the two types of fibers are overlapping or the cumulative distribution is continuous. The presence of discontinuity modifies the avalanche size distribution for smaller Δ [22]. Hence one expects the mean field (GLS) behavior of the mixed model. The behavior of the imminent failure shows a crossover in the avalanche size exponent from $5/2$ to $3/2$ as the critical distribution is approached. In some distributions (depending upon ρ_1 and ρ_2), the effective critical stress is found to vary linearly with the mixing parameter x. We have pointed out the origin of the apparent linear behavior and argued that in general a nonlinear variation is expected.

ACKNOWLEDGMENTS

We sincerely thank Y. Moreno, A. F. Pacheco, and S. Pradhan for their help and B. K. Chakrabarti for useful comments.

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